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Quantum treatment of a class of time-dependent potentials

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Abstract. The time-dependent potential $V(x-f(t))$ is studied by path integrals. It is shown that the problem can be mapped into the static form of the potential plus a linear term with a time-dependent coefficient. After the presentation of the general formulation, some exactly solvable examples are discussed. A perturbative treatment is also suggested.

1. Introduction

Quantisation of time-dependent dynamical systems is an attractive problem of theoretical physics. We can classify these systems into two main categories, the first of which consists of the quantum mechanical particle motions which are subject to time-dependent boundary conditions [1].

Problems of time-dependent potentials with fixed boundary conditions belong to the second category. Among them we can mention the quadratic potential with time-dependent coefficients [2, 3]. This problem has also been studied by path integrals [4].

In this paper we study a particular type of time-dependent potential

$$V(x, t) = V(x-f(t)) \tag{1}$$

which is obtained by translating the argument of $V(x)$ by an arbitrary function $f(t)$. To solve this problem we transform to new coordinates in which the potential consists of two terms: a fixed form of the original potential and a linear term with a time-dependent coefficient. Separation of the fixed and the time-dependent parts of the potential makes it possible to get exact solutions for some particular examples. It also permits one to treat the time-dependent term as a perturbation.

In this paper we will employ the path integration method. Since it deals with classical phase space, path integration is much suited to the application of required time-dependent canonical transformations.

2. Formulation of the problem with path integrals

In this section we will present a general treatment of the moving potential problem of equation (1). The probability amplitude for the motion of a particle with mass μ, from spacetime point x_a, t_a to x_b, t_b under the influence of potential $V(x, t)$ can be written as a phase-space path integral:

$$K(x_b, t_b; x_a, t_a) = \int \mathscr{D}x\mathscr{D}p_x \exp\left[\mathrm{i}\int_{t_a}^{t_b} \mathrm{d}t \left(p_x\dot{x} - \frac{p_x^2}{2\mu} - V(x-f(t))\right)\right] \tag{2}$$

where the overdot stands for $\mathrm{d}/\mathrm{d}t$. This expression is understood as the limit of the usual time-graded formula:

$$K(x_b, t_b; x_a, t_a) = \lim_{\substack{n\to\infty \\ \varepsilon\to 0}} \prod_{j=1}^{n} \int \mathrm{d}x_j \prod_{j=1}^{n+1} \int \frac{\mathrm{d}p_{x_j}}{2\pi} \prod_{j=1}^{n+1}$$

$$\times \exp\left[\mathrm{i}\left(p_{x_j}(x_j - x_{j-1}) - \frac{\varepsilon p_{x_j}^2}{2\mu} - \varepsilon V(x_j - f(t_j))\right)\right] \tag{3}$$

with

$$t_a = t_0 \qquad t_b = (n+1)\varepsilon + t_0 = t_{n+1}$$

and

$$x_a = x_0 \qquad x_b = x_{n+1}.$$

In order to map the problem to a familiar form of a time-independent potential, we apply a time-dependent translation to x by $x \to x + f(t)$. By explicitly transforming the action we observe that the kinetic energy terms of the resulting Hamiltonian becomes $(1/2\mu)(p_x - \mu\dot{f})^2$ which suggests a translation of momentum by $p_x \to p_x + \mu\dot{f}$. These translations in coordinate and momentum variables can be combined into a time-dependent canonical transformation:

$$Q = x - f(t) \qquad P = p_x - \mu\dot{f}(t) \tag{4}$$

generated by

$$F_2(x, P, t) = (P + \mu\dot{f})(x - f). \tag{5}$$

The Hamiltonian and the action become

$$H_Q = H_x + \frac{\partial F_2}{\partial t}$$

$$= \frac{1}{2\mu} P^2 + V(Q) + \mu\ddot{f}Q - \tfrac{1}{2}\mu\dot{f}^2 \tag{6}$$

and

$$\int_{t_a}^{t_b} \mathrm{d}t(p_x\dot{x} - H_x) = \int_{t_a}^{t_b} \mathrm{d}t\left(-Q\dot{P} - H_Q + \frac{\mathrm{d}}{\mathrm{d}t} F_2\right)$$

$$= \mu(\dot{f}(t_b)Q_b - \dot{f}(t_a)Q_a) + \int_{t_a}^{t_b} \mathrm{d}t(P\dot{Q} - H_Q). \tag{7}$$

In developing the last equation we have used

$$\frac{\mathrm{d}}{\mathrm{d}t} F_2 = \frac{\mathrm{d}}{\mathrm{d}t}(P + \mu\dot{f})Q$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(PQ) + \mu\frac{\mathrm{d}}{\mathrm{d}t}(\dot{f}Q).$$

By the help of (6) and (7), the amplitude of (2) can be written as

$$K(x_b, t_b; x_a, t_a) = \exp\{\mathrm{i}\mu[\dot{f}_b(x_b - f_b) - \dot{f}_a(x_a - f_a)]\}$$

$$\times \exp\left(\mathrm{i}\frac{\mu}{2}\int_{t_a}^{t_b} \mathrm{d}t\dot{f}^2\right) \bar{K}(Q_b, t_b; Q_a, t_a) \tag{8}$$

where

$$\bar{K}(Q_b, t_b; Q_a, t_a) = \int \mathscr{D}Q\mathscr{D}P \exp\left[\mathrm{i}\int_{t_a}^{t_b} \mathrm{d}t\left(P\dot{Q} - \frac{P^2}{2\mu} - V(Q) - \mu\ddot{f}Q\right)\right]. \tag{9}$$

The price paid for removing the translational time dependence of the original potential is the introduction of the extra term $\mu\ddot{f}Q$, which represents a homogeneous time-dependent force. The situation can be compared with the mapping of a time-dependent boundary condition problem into the time-dependent harmonic oscillator potential [1].

If the problem can be exactly treated for the static potential $V(Q)$, then we can solve the path integration of (9) in the form of series by regarding $\mu\ddot{f}Q$ as a time-dependent perturbation.

Inspecting the result of (9) we notice that, for translations linear or quadratic in time, the potential in Q-space is stationary.

For potentials whose sources are moving with constant acceleration, that is for

$$f(t) = \tfrac{1}{2}\gamma t^2 \qquad \gamma = \text{constant} \tag{10}$$

equation (8) becomes

$$\begin{aligned} K(x_b, t_b; x_a, t_a) &= \exp\{\mathrm{i}\mu[\gamma(x_b t_b - x_a t_a) - \tfrac{1}{3}\gamma^3(t_b^3 - t_a^3)]\} \\ &\quad\times\int \mathscr{D}Q\mathscr{D}P \exp\left[\mathrm{i}\int_{t_a}^{t_b}\mathrm{d}t\left(P\dot{Q} - \frac{P^2}{2\mu} - V(Q) - \mu\gamma Q\right)\right]. \end{aligned} \tag{11}$$

As an exactly solvable example consider an oscillator potential

$$V(x, t) = \tfrac{1}{2}\mu\omega^2(x - \tfrac{1}{2}\gamma t^2)^2. \tag{12}$$

For this case the potential in Q-space is

$$\begin{aligned} U(Q) &= \tfrac{1}{2}\mu\omega^2 Q^2 - \mu\gamma Q \\ &= \tfrac{1}{2}\mu\omega^2(Q + \gamma/\omega^2)^2 + \tfrac{1}{2}\mu(\gamma/\omega^2)^2. \end{aligned} \tag{13}$$

Inserting the well known Green function formula for the above potential [5] into (11) we have

$$\begin{aligned} K(x_b, t_b; x_a, t_a) &= \exp\left[\mathrm{i}\mu\left(\gamma(x_b t_b - x_a t_a) - \frac{\gamma^2}{3}(t_b^3 - t_a^3) + \frac{\gamma^2}{2\omega^4}(t_b - t_a)\right)\right] \\ &\quad\times\left(\frac{\mu\omega}{2\pi\mathrm{i}\sin\omega T}\right)^{1/2}\exp\left[\!\left[\frac{\mathrm{i}\mu\omega}{2\sin\omega T}\left\{\left[\left(x_a - \frac{\gamma}{2}t_a^2 + \frac{\gamma}{\omega^2}\right)^2\right.\right.\right.\right. \\ &\quad\left. + \left(x_b - \frac{\gamma}{2}t_b^2 + \frac{\gamma}{\omega^2}\right)^2\right]\cos\omega T \\ &\quad\left.\left. - 2\left(x_a - \frac{\gamma}{2}t_a^2 + \frac{\gamma}{\omega^2}\right)\left(x_b - \frac{\gamma}{2}t_b^2 + \frac{\gamma}{\omega^2}\right)\right\}\right]\!\right] \end{aligned} \tag{14}$$

where $T = t_b - t_a$. Using the usual expansion of the oscillator Green function in terms of the Hermite polynomials [5] we obtain the wavefunction for the potential of (12):

$$\psi_\gamma(x,t) = \exp\left\{-\mathrm{i}\left[\omega\left(n+\frac{1}{2}\right) - \mu\gamma\left(x+\frac{\gamma}{2\omega^4}\right)\right]t\right\}\exp\left(-\mathrm{i}\mu\frac{\gamma^2}{3}t^3\right)\left(\frac{(\mu\omega)^{1/2}}{\pi^{1/2}2^n n!}\right)^{1/2}$$
$$\times H_n\left[\sqrt{\mu\omega}\left(x-\frac{1}{2}\gamma t^2+\frac{\gamma}{\omega^2}\right)\right]\exp\left[-\frac{\mu\omega}{2}\left(x-\frac{1}{2}\gamma t^2+\frac{\gamma}{\omega^2}\right)^2\right]. \qquad (15)$$

The second and more simple case occurs for

$$f(t) = vt \qquad v = \text{constant} \qquad (16)$$

in which, as the natural consequence of the Galilean invariance of Schrödinger equation, the potential in Q-space does not acquire any additional term. Then the amplitude becomes

$$K(x_b, t_b; x_a, t_a)$$
$$= \exp\left[-\tfrac{1}{2}\mathrm{i}\mu v^2(t_b - t_a) + \mathrm{i}\mu v(x_b - x_a)\right]$$
$$\times \int \mathscr{D}Q\mathscr{D}P \exp\left[\mathrm{i}\int_{t_a}^{t_b} \mathrm{d}t\left(P\dot{Q} - \frac{P^2}{2\mu} - V(Q)\right)\right]. \qquad (17)$$

From this formula we conclude that, if the solution of the Schrödinger equation for a stationary potential $V(x)$ is $\psi(x,t) = \exp(\mathrm{i}Et)\Phi(x)$ the wavefunction for $V(x-vt)$ is given by

$$\psi_v(x,t) = \exp[-\mathrm{i}(E + \tfrac{1}{2}\mu v^2)t]\exp(\mathrm{i}\mu vx)\Phi(x - vt). \qquad (18)$$

As a specific example for the linear translations we consider a δ-function potential:

$$V(x,t) = -\frac{\alpha}{2\mu}\delta(x - vt). \qquad (19)$$

For $\alpha > 0$, the potential is attractive and there exists one bound state. To obtain this state we insert the known result for the stationary wavefunction [6]

$$\psi(Q,t) = \left(\frac{\alpha}{2}\right)\exp\left(\mathrm{i}\frac{\alpha^2}{8\mu}t\right)\exp\left(-\frac{\alpha}{2}|Q|\right) \qquad (20)$$

into (18) and arrive at

$$\psi_v(x,t) = \left(\frac{\alpha}{2}\right)\exp\left[\frac{\mathrm{i}}{2\mu}\left(\frac{\alpha^2}{4} - \mu^2 v^2\right)t\right]\exp\left(\mathrm{i}\mu vx - \frac{\alpha}{2}|x - vt|\right). \qquad (21)$$

For a fixed position $x > 0$, and for $v > 0$, if the time is early enough to satisfy

$$t < x/v$$

then the wavefunction of (21) is

$$\psi_v^{\text{early}}(x,t) = \left(\frac{\alpha}{2}\right)^{1/2}\exp\left[\frac{\mathrm{i}}{2\mu}\left(\frac{\alpha}{2} - \mathrm{i}\mu v\right)^2 t\right]\exp\left[-\left(\frac{\alpha}{2} - \mathrm{i}\mu v\right)x\right] \qquad (22)$$

whose magnitude increases by $\exp(\frac{1}{2}\alpha vt)$.

For later times, i.e. for

$$x - vt < 0$$

the wavefunction becomes

$$\psi_v^{\text{late}}(x,t)=\left(\frac{\alpha}{2}\right)^{1/2}\exp\left[\frac{\mathrm{i}}{2\mu}\left(\frac{\alpha}{2}+\mathrm{i}\mu v\right)^2 t\right]\exp\left[\left(\frac{\alpha}{2}+\mathrm{i}\mu v\right)x\right] \tag{23}$$

which decays by the factor $\exp(-\frac{1}{2}\alpha v t)$.

At this point we emphasise that the case given by (16) occurs in many problems of electrodynamics involving vector potentials. For example, consider the interaction of a light wave of linear polarisation, propagating in the x direction, with an electron [7]. The light wave can be described by the vector potential

$$A_y=\frac{\mathscr{E}_0}{\omega}\cos(\omega t-kx) \qquad A_x=A_y=0. \tag{24}$$

The amplitude for this problem is

$$K(\boldsymbol{x}_b,t_b;\boldsymbol{x}_a,t_a)=\int \mathscr{D}^3x\mathscr{D}^3p\exp\left[\mathrm{i}\int_{t_a}^{t_b}\mathrm{d}t\left(\boldsymbol{p}\cdot\dot{\boldsymbol{x}}-\frac{\boldsymbol{p}^2}{2\mu}-\frac{e\mathscr{E}_0}{\omega}\dot{y}\cos(\omega t-kx)\right)\right]. \tag{25}$$

The explicit definition of the above formula can be obtained by obvious generalisation of (3) to three dimensions. By translating the variable p_y as

$$p_y\to p_y+\frac{e\mathscr{E}_0}{\omega}\cos(\omega t-kx)$$

equation (25) can be written as

$$\begin{aligned}K(\boldsymbol{x}_b,t_b;\boldsymbol{x}_a,t_a)&\\ =\int \mathscr{D}^3x\mathscr{D}^3p\exp&\left[\mathrm{i}\int_{t_a}^{t_b}\mathrm{d}t\left(p_x\dot{x}+p_y\dot{y}+p_z\dot{z}-\frac{p_x^2+p_y^2+p_z^2}{2\mu}\right.\right.\\ &\left.\left.-\frac{e\mathscr{E}_0}{\mu\omega}p_y\cos(\omega t-kx)-\frac{e^2\mathscr{E}_0^2}{2\mu\omega^2}\cos^2(\omega t-kx)\right)\right].\end{aligned} \tag{26}$$

Here the functional integrations in the y and z directions are trivial which result in fixing p_y and p_z to be the constant momentum components:

$$\begin{aligned}K(\boldsymbol{x}_b,t_b;\boldsymbol{x}_a,t_a)=&\int\frac{\mathrm{d}p_y\,\mathrm{d}p_z}{(2\pi)^2}\exp[\mathrm{i}p_y(y_b-y_a)+\mathrm{i}p_z(z_b-z_a)]\\ &\times\exp\left(-\mathrm{i}\frac{p_y^2+p_z^2}{2\mu}(t_b-t_a)\right)K_{p_y}(x_b,t_b;x_a,t_a).\end{aligned} \tag{27}$$

K_{p_y} is defined as

$$\begin{aligned}K_{p_y}(x_b,t_b;x_a,t_a)&\\ =\int\mathscr{D}x\mathscr{D}p_x\exp&\left[\mathrm{i}\int_{t_a}^{t_b}\mathrm{d}t\left(p_x\dot{x}-\frac{p_x^2}{2\mu}-\frac{e\mathscr{E}_0}{\mu\omega}p_y\cos(\omega t-kx)\right.\right.\\ &\left.\left.-\frac{e^2\mathscr{E}_0^2}{2\mu\omega^2}\cos^2(\omega t-kx)\right)\right]\end{aligned} \tag{28}$$

which is of the type that we are interested in. Employing the canonical transformation of (4) with $f(t)=(\omega/k)t$ the potential in Q-space becomes

$$V(Q)=\frac{e\mathscr{E}_0}{\mu\omega}p_y\cos kQ+\frac{e^2\mathscr{E}_0^2}{2\mu\omega_0^2}\cos^2 kQ. \tag{29}$$

If one of the terms in this potential is negligible (for example, if $\mathscr{E}_0 \ll \omega$ or $\mathscr{E}_0 \gg \omega$) the corresponding quantum mechanical problem is exactly solvable in terms of the Mathieu functions [7].

For frequencies depending on time, i.e. for $\omega = \omega(t)$, the problem becomes complicated and the additional linear term $(\mu/k)(\ddot{\omega}t + 2\dot{\omega})Q$ that (29) would acquire should be treated as a perturbation.

As the final example we wish to remark that our method suggests an elegant procedure, which is alternative to the usual one [4], for the path integration of the potential

$$V(x, t) = ax^2 + bt^n x \tag{30}$$

with $a, b = \text{constant}$ and $n = \text{integer}$. We first complete the squares and obtain

$$V(x, t) = a\left(x + \frac{b}{2a}t^n\right)^2 - \frac{b^2}{4a^2}t^{2n}$$

where the last term can be integrated out of the action. After translating x by

$$x \to Q = x + \frac{b}{2a}t^n$$

we arrive at a problem in Q-space with potential

$$V(Q, t) = aQ^2 + \frac{\mu b}{2a}n(n-1)t^{n-2}Q. \tag{31}$$

Repeating the same procedure $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$ times for $n = \text{even}$ or $n = \text{odd}$, respectively, we have a time-independent problem.

3. Discussion

We have demonstrated that the potential

$$V(x - f(t))$$

can be mapped into

$$U(Q) = V(Q) + \mu\frac{d^2 f}{dt^2}Q$$

which is the static form of the original potential plus a linear term with a time-dependent coefficient. One advantage of this mapping is that if the quantum mechanical problem for $V(Q)$ is exactly solvable, we can write the solution for the full problem in the series form by regarding the time-dependent linear term as a perturbation. In that respect our problem closely resembles the time-dependent boundary condition problem [1], where the original Schrödinger equation is transformed into a time-dependent oscillator potential problem in fixed boundaries, which is treated as a perturbation.

Perturbations can be either performed in the Schrödinger picture as done in [1], or in the path integral formulation. A brief outline of the path integral approach is as follows.

We rewrite (9) by expanding the perturbative term of the action into the power series:

$$\bar{K}(b, a) = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \mathscr{D}Q\mathscr{D}P \left(\int_{t_a}^{t_b} \mathrm{d}t \mu \ddot{f} Q\right)^n \times \exp\left[\mathrm{i}\int_{t_a}^{t_b} \mathrm{d}t \left(P\dot{Q} - \frac{P^2}{2\mu} - V(Q)\right)\right] \tag{32}$$

with b, a standing for the final and initial spacetime points. After the familiar manipulations we arrive at [5]:

$$\bar{K}(b, a) = K_0(b, a) + K^{(1)}(b, a) + K^{(2)}(b, a) + \ldots \tag{33}$$

where

$$K_0(b, a) = \int \mathscr{D}Q\mathscr{D}P \exp\left[\mathrm{i}\int_{t_a}^{t_b} \mathrm{d}t \left(P\dot{Q} - \frac{P^2}{2\mu} - V(Q)\right)\right]$$

is the propagator for potential $V(Q)$, whereas the following terms represent propagations with several number of interactions via the perturbative term. For example,

$$K^{(1)}(b, a) = -\mathrm{i}\int_{t_a}^{t_b} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}Q_1 K_0(Q_b, t_b; Q_1, t_1)\mu\ddot{f}(t_1)Q_1 K_0(Q_1, t_1; Q_a, t_a)$$

is the propagator from point a to b with one scattering at the spacetime point Q_1, t_1. From one interaction point to the next one the system propagates by the Green function of the static potential $V(Q)$. Examples of the perturbative path integrations can be found in [5], and in the recent formulation of QED in terms of the classical particle trajectories [8].

Finally, we wish to point out that there exist some simpler and exactly solvable cases. (i) When $f(t)$ is a linear or quadratic function of t the potentials in Q-space are time independent. (ii) The oscillator potential problem is exactly solvable for all forms of $f(t)$ since the exact solutions of $V = x^2 + g(t)x$ are available with $g(t)$ being an arbitrary function of time [5].

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